How can we find the 2D-transformation between 2 point sets and get the scale, rotation, shear, and offset?

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Suppose we have 2 sets of points called source set and target set where each point from one set corresponds to exactly one point from the other set. We need to find the matrix (M) which will transform the points from the source set so that the distance between each point and its corresponding point in the target set is minimized.

Linear Least Square

The simplest way is to solving the least square problem of the form $\overline{X} = MX$ where X is the set of points from the source set, $\overline{X}$ is the set of points from the target, M is the 2D-transformation matrix. X and $\overline{X}$ are 3xN matrices where each column is the coordinate of each point of the form $\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$.

The matrix M has the following form:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

M is actually an augmentation of a matrix 2D-affine matrix A with the translation vector T add on the rightmost column. The affine matrix and the translation matrix has the form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Which allows us to write M as:

$$M = \begin{bmatrix} A & T \\ 0 & 1 \end{bmatrix}$$

This form has the advantage that we can compose different transformations simply by multiplying the matrices. With this form we can also simultaneously estimate A and T by solving for M.

How can we extract scale, rotation, shear, and offset from M?

Once we obtain M, we can extract these from the M by decomposing it into several transformation matrices.

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \Lambda & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}$$

Here S is the scaling matrix, Λ is shear matrix, R is the rotation matrix, and T is the translation vector. Thus we can interpret the matrix M to be the composition of scaling operation, followed by shear, then rotation aroung the origin, and then translation.

We can obtain the following equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} * \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} * \begin{bmatrix} s_x & ms_y \\ 0 & s_y \end{bmatrix}$$

Then we can simplify it further:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s_x cos(\theta) & s_y mcos(\theta) - s_y sin(\theta) \\ s_x sin(\theta) & s_y msin(\theta) + s_y cos(\theta) \end{bmatrix}$$

Translation

The translations are easy to extract, we can simply find it as t_x and t_y from M.

Rotation

$$\begin{aligned} a &= s_x cos(\theta) \\ c &= s_x sin(\theta) \\ c/a &= tan(\theta) \\ &\boxed{\theta = tan^{-1}(\frac{c}{a})} \end{aligned}$$

Shear

$$\begin{aligned} b &= s_y(mcos(\theta) - sin(\theta)) \\ d &= s_y(msin(\theta) + cos(\theta)) \end{aligned}$$

We can get rid of s_y as follows:

$$\begin{aligned} \frac{b}{mcos(\theta) - sin(\theta)} &= \frac{d}{msin(\theta) + cos(\theta)} \\ m(bsin(\theta) - dcos(\theta)) &= -dsin(\theta) - bcos(\theta) \\ &\boxed{m = \frac{dsin(\theta) + bcos(\theta)}{dcos(\theta) - bsin(\theta)}} \end{aligned}$$

Scales

Horizontal scale s_x can be derived as follows:

$$
\begin{aligned}
a &= s_x cos(\theta) \\
c &= s_x sin(\theta) \\
a^2 + c^2 &= s_x^2 cos(\theta)^2 + s_x^2 sin^2(\theta) \\
a^2 + c^2 &= s_x^2 (cos(\theta)^2 + sin^2(\theta)) \\
a^2 + c^2 &= s_x^2 \\
&\boxed{s_x = \pm\sqrt{a^2 + c^2}}
\end{aligned}
$$

If we would like to know the scale as well the sign of the scale, we can also obtain s_x by computing :

$$\boxed{s_x = \frac{a}{cos(\theta)}}$$

or

$$\boxed{s_x = \frac{c}{sin(\theta)}}$$

The vertical scale s_y is derived as follows:

$$
\begin{aligned}
b &= s_y(mcos(\theta) - sin(\theta)) \\
d &= s_y(msin(\theta) + cos(\theta))
\end{aligned}
$$

Since m and θ are already known, we can simply compute s_y with:

$$\boxed{s_y = \frac{b}{mcos(\theta) - sin(\theta)}}$$

$$or$$

$$\boxed{s_y = \frac{d}{msin(\theta) + cos(\theta)}}$$